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Algebraic theories of compact pospaces

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Abstract

Let **CmptPoSp** denote the category of compact pospaces with continuous monotone maps and let **PoSet** denote the category of partially ordered sets and monotone maps. In this paper we show that the forgetful functor $G:\mathbf{CmptPoSp} \to \mathbf{PoSet}$ is monadic; that is, G has a left-adjoint and **CmptPoSp** is isomorphic to the category of algebras $\mathbf{PoSet}^{\boldsymbol{B}}$ for the monad $\boldsymbol{B}$ on **PoSet** induced by the adjunction. This result, which is an asymmetric version of Manes' theorem, shows that the notion of compact pospace is algebraic in a precise sense and provides a useful tool for investigating the category **CmptPoSp**. As a corollary we obtain the theorem of Simmons and Wyler which says that **CmptPoSp** is also algebraic over the category of topological spaces and continuous maps. This makes explicit the connection between the Salbany and the prime Wallman compactifications. We also give an explicit construction—as the prime spectrum of the lattice of upper sets—of the Stone–Čech–Nachbin order compactification for a discrete ordered space. © 1997 Elsevier Science B.V.

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1. Introduction

A *compact pospace* consists of a compact topological space (X,τ) together with a partial ordering $\leqslant$ of X which is closed in the product space $(X,\tau)\times(X,\tau)$. These spaces were introduced by Nachbin [9] and recent results [4,3,5,12] have shown that they provide the appropriate analogue in asymmetric topology of compact Hausdorff spaces.

Let **CmptPoSp** denote the category of compact pospaces with continuous monotone maps and let **PoSet** denote the category of partially ordered sets and monotone maps. In

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this paper we show that the forgetful functor $G : \mathbf{CmptPoSp} \to \mathbf{PoSet}$ is *monadic*; that is, G has a left-adjoint and **CmptPoSp** is isomorphic to the category of algebras $\mathbf{PoSet}^{\boldsymbol{B}}$ for the monad $\boldsymbol{B}$ on **PoSet** induced by the adjunction. This result, which is an asymmetric version of Manes' theorem [6], shows that the notion of compact pospace is algebraic in a precise sense and provides a useful tool for investigating the category **CmptPoSp**. As a corollary we obtain the theorem of Simmons [11] and Wyler [14] which says that **CmptPoSp** is also algebraic over the category of topological spaces and continuous maps. This makes explicit the connection between the Salbany [10] and the prime Wallman [14] compactifications. We also give an explicit construction—as the prime spectrum of the lattice of upper sets—of the Stone–Čech–Nachbin order compactification for a discrete ordered space.

The category of posets cannot be replaced by the category of sets in our main result. To see this, recall that by Beck's theorem [7, §3.1] a monadic functor $G : \boldsymbol{A} \to \boldsymbol{X}$ *reflects isomorphisms*: if $f : a \to a'$ is a morphism in $\boldsymbol{A}$ such that $Gf : Ga \to Ga'$ is an isomorphism in $\boldsymbol{X}$, then f must be an isomorphism in $\boldsymbol{A}$. Let τ denote the usual topology on the unit interval, $\mathbb{I}$, and let $\leqslant$ denote the usual ordering on $\mathbb{I}$. Then the identity map $\mathrm{Id} : (\mathbb{I}, \tau, =) \to (\mathbb{I}, \tau, \leqslant)$ is continuous and monotone and, regarded as a map of the underlying sets, is an isomorphism. But $\mathrm{Id} : (\mathbb{I}, \tau, =) \to (\mathbb{I}, \tau, \leqslant)$ is not an isomorphism of partially ordered topological spaces. Consequently, the forgetful functor from compact pospaces to the category of sets does not reflect isomorphisms and so is not monadic.

2. Preliminaries

Let **DLat** denote the category of (bounded) distributive lattices and lattice homomorphisms and let **Sp** denote the category of T_0 topological spaces and continuous maps. Recall that a *filter* on a lattice L is a nonempty subset $\mathcal{F}$ of L satisfying

– if $a \in \mathcal{F}$ and $a \leqslant b$, then $b \in \mathcal{F}$; and
– if $a \in \mathcal{F}$ and $b \in \mathcal{F}$, then $a \wedge b \in \mathcal{F}$.

A filter $\mathcal{F}$ is *proper* if $\mathcal{F} \neq L$ and *prime* if it is proper and whenever $a \vee b \in \mathcal{F}$, either $a \in \mathcal{F}$ or $b \in \mathcal{F}$. Let $\mathtt{Spec}(L)$ denote the collection of prime filters on L. The *spectral topology* on $\mathtt{Spec}(L)$ is that with basis the family of sets of the form $a^{\#} = \{\mathcal{F} \in \mathtt{Spec}(L) \mid a \in \mathcal{F}\}$, for $a \in L$. The *patch topology* on $\mathtt{Spec}(L)$ is that with basis the family of sets of the form $a^{\#} \setminus b^{\#}$ for $a, b \in L$. $\mathtt{Spec}$ extends to a contravariant functor $\mathtt{Spec} : \mathbf{DLat} \to \mathbf{Sp}^{\mathrm{op}}$ which sends a distributive lattice L to the space of prime filters on L with the spectral topology and sends a lattice homomorphism $f : L \to M$ to the map $\mathtt{Spec}(f) : \mathtt{Spec}(M) \to \mathtt{Spec}(L)$, where $\mathtt{Spec}(f)(\mathcal{F}) = f^{-1}(\mathcal{F})$. Since $\mathtt{Spec}(f)^{-1}(a^{\#}) = (f(a))^{\#}$, for $a \in L$, $\mathtt{Spec}(f)$ is continuous. We note for later reference that $\mathtt{Spec}(f)$ is also continuous for the patch topologies and monotone, if we partially order the prime filters by inclusion.

For a set X, we write $\beta(X, =)$ for $\mathtt{Spec}\,\mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X. Thus $\beta(X, =)$ is the set of ultrafilters on a set X with the *hull-kernel topology*, which is, of course, just the Stone–Čech compactification of the discrete space X. To avoid confusion below, we denote the basic open subset corresponding to $Z \subseteq X$ by $Z^{\sharp,=} = \{\mathcal{F} \in \beta(X,=) \mid Z \in \mathcal{F}\}$. We will need the following well known facts about the spectral topology on $\beta(X,=)$.

Proposition 1. *$\beta(X,=)$ is compact, Hausdorff and zero-dimensional.*

Proof. Since $\beta(X,=)\backslash Z^{\sharp,=} = (X\backslash Z)^{\sharp,=}$, each basic open set $Z^{\sharp,=}$ is clopen, so $\beta(X,=)$ is zero-dimensional. Suppose that $\{Z^{\sharp,=}\}_{Z\in\mathcal{A}}$ is a family of basic closed sets with the f.i.p. Then $\{Z\}_{Z\in\mathcal{A}}$ is a family of subsets of X with the f.i.p. and we can choose $\mathcal{F} \in \beta(X,=)$ so that $\mathcal{A} \subseteq \mathcal{F}$. Then $\mathcal{F} \in \bigcap_{Z\in\mathcal{A}} Z^{\sharp,=}$. Finally, if $\mathcal{F} \neq \mathcal{G}$, let $Z \in \mathcal{F}\backslash\mathcal{G}$. Then $\mathcal{F} \in Z^{\sharp,=}$, $\mathcal{G} \in (X\backslash Z)^{\sharp,=}$ and $Z^{\sharp,=}$ and $(X\backslash Z)^{\sharp,=}$ are disjoint open sets. □

For a poset $(X,\leqslant)$, we will use the following notation and terminology. For $A \subseteq X$,

$$\uparrow A = \{x \in X \mid \text{for some } a \in A,\ a \leqslant x\} \quad \text{and}$$
$$\downarrow A = \{x \in X \mid \text{for some } a \in A,\ x \leqslant a\}.$$

For $a \in X$, $\uparrow a = \uparrow\{a\}$ and $\downarrow a = \downarrow\{a\}$. A is an *upper* (*lower*) *set* if $\uparrow A = A$ ($\downarrow A = A$). Finally, let $\alpha(X,\leqslant)$ denote the collection of all upper subsets of X.

Assume (X,τ) is a topological space. For $x \in X$, we let $\mathcal{N}(x) = \{U \in \tau \mid x \in U\}$ denote the neighborhood filter of x. The *specialization order* of (X,τ), denoted by $\leqslant_\tau$, is defined by $x \leqslant_\tau y$ iff for all $U \in \tau$, if $x \in U$, then $y \in U$. Thus $x \leqslant_\tau y$ iff $x \in \mathrm{cl}_\tau\{y\}$.

By a *partially ordered topological space* [9] we understand a triple $(X,\tau,\leqslant)$, where (X,τ) is a topological space and $\leqslant$ is a partial ordering on X. If $(X,\tau,\leqslant)$ is a partially ordered topological space, we let $\tau^{\uparrow} = \{U \in \tau \mid U = \uparrow U\}$ and $\tau^{\downarrow} = \{U \in \tau \mid U = \downarrow U\}$. Clearly $\tau^{\uparrow}$ and $\tau^{\downarrow}$ are topologies on X.

Definition 2. Assume that $(X,\tau,\leqslant)$ is a partially ordered topological space.

(1) $(X,\tau,\leqslant)$ is a pospace [1, §VI.1] if $\leqslant$ is closed as a subset of $(X,\tau)\times(X,\tau)$.

(2) $(X,\tau,\leqslant)$ is totally order-separated if whenever $x \nleqslant y$ there is an clopen upper set U such that $x \in U$ and $y \notin U$.

(3) $(X,\tau,\leqslant)$ is a Priestley space if it is totally order-separated and (X,τ) is compact.

Evidently, every Priestley space is a pospace and every pospace is Hausdorff.

Proposition 3 (cf. [3, Lemma 1]). *Assume that $(X,\tau,\leqslant)$ is a compact pospace. For all $x \in X$ and all $\mathcal{F} \in \beta(X,=)$, $\mathcal{F} \to_{\tau^\uparrow} x$ iff $x \leqslant \lim_\tau \mathcal{F}$.*

Proof. Suppose $x \leqslant \lim_\tau \mathcal{F}$. If $x \in U \in \tau^{\uparrow}$, then $\lim_\tau \mathcal{F} \in U \in \tau$ and so $U \in \mathcal{F}$. Thus $\mathcal{F} \to_{\tau^\uparrow} x$.

Suppose $\mathcal{F} \to_{\tau^\uparrow} x$. Let $y = \lim_\tau \mathcal{F}$ and suppose $x \nleqslant y$. By Proposition VI.1.6(ii) of [1], $\uparrow(x)$ and $\downarrow(y)$ are closed subsets of X. Since $\uparrow(x)$ is an upper set and $\downarrow(y)$ is a lower set and they are disjoint, by VI.1.8 of [1], there is an open upper set U and an open lower set V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. But then $U, V \in \mathcal{F}$, which is absurd. Thus $x \leqslant \lim_\tau \mathcal{F}$ $\square$

For basic concepts of category theory we refer to MacLane [8], especially the first three sections of Chapter VI for an introduction to monads and algebras for a monad. The only concept we will need beyond what is treated there is that of a morphism of monads.

Definition 4. Assume that $\boldsymbol{T} = (T, \eta, \mu)$ and $\boldsymbol{T} = (T', \eta', \mu')$ are monads on the categories $\mathcal{A}$ and $\mathcal{A}'$, respectively. Then a morphism $(R, \pi) : \boldsymbol{T} \to \boldsymbol{T}$ consists of a functor $R : \mathcal{A} \to \mathcal{A}'$ and a natural transformation $\pi : T'R \to RT$ such that for all objects A of $\mathcal{A}$ the following two diagrams commute:

$$\begin{tikzcd} RA \arrow[r, "\eta'_{RA}"] \arrow[dr, "R\eta_A"'] & T'RA \arrow[d, "\pi_A"] \\ & RTA \end{tikzcd} \qquad \begin{tikzcd} T'^2RA \arrow[rr, "\mu'_{RA}"] \arrow[d, "T'\pi_A"'] & & T'RA \arrow[d, "\pi_A"] \\ T'RTA \arrow[r, "\pi_{TA}"'] & RT^2A \arrow[r, "R\mu_A"'] & RTA \end{tikzcd}$$

A morphism $(R, \pi) : \boldsymbol{T}' \to \boldsymbol{T}$ of monads induces a functor $(R, \pi)^\star : \mathcal{A}^{\boldsymbol{T}} \to \mathcal{A}'^{\boldsymbol{T}'}$ between the categories of algebras as follows: for a $\boldsymbol{T}$-algebra (A, α), $(R, \pi)^\star(A, \alpha) = (RA, R\alpha \circ \pi_A)$ and for $f : (A, \alpha) \to (B, \beta)$ a morphism of $\boldsymbol{T}$-algebras, $(R, \pi)^\star(f) = Rf$. This functor is *algebraic*; that is, it makes the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[d, "R"'] & \mathcal{A}^{\boldsymbol{T}} \arrow[l, "G^{\boldsymbol{T}}"'] \arrow[d, "(R,\pi)^\star"] \\ \mathcal{A}' & \mathcal{A}'^{\boldsymbol{T}} \arrow[l, "G^{\boldsymbol{T}}"] \end{tikzcd}$$

commute. In fact, any algebraic functor $R^\sharp : \mathcal{A}^{\boldsymbol{T}} \to \mathcal{A}'^{\boldsymbol{T}'}$ is induced by a morphism of monads in this way [13].

3. The prime upper filter monad

We begin this section by finding a suitable analog for the space of ultrafilters (the Stone–Čech compactification) when a discrete set is replaced by a discrete partially ordered space. For this, we replace the power set by the set of all upper sets and ultrafilters by prime filters. If the order is equality, then the collection of all upper subsets is just the power set and a filter is prime iff it is an ultrafilter, so, in this case, the two constructions coincide.

Recall that for a poset $(X, \leqslant)$, $\alpha(X, \leqslant)$ denotes the collection of all upper subsets of X. Let $\beta(X, \leqslant)$ denote $\mathtt{Spec}\,\alpha(X, \leqslant)$ with the patch topology and partially ordered by set inclusion. Thus $\beta(X, \leqslant)$ is the collection of prime filters on αX. For $Z \in \alpha X$, we denote the corresponding basic open subset in the spectral topology by $Z^{\sharp,\leqslant} = \{\mathcal{F} \in$

$\beta(X,\leqslant) \mid Z \in \mathcal{F}\}$. The patch topology has as base for the opens all sets of the form $Z^{\sharp,\leqslant}\backslash W^{\sharp,\leqslant}$, for $Z, W \in \alpha X$.

For $\mathcal{F} \in \beta(X,=)$, let $\mathcal{F}^{\uparrow} = \{A \in \mathcal{F} \mid A = {\uparrow}A\}$ and $\mathcal{F}^{\downarrow} = \{A \in \mathcal{F} \mid A = {\downarrow}A\}$. Clearly, if $\mathcal{F} \in \beta(X,=)$, then $\mathcal{F}^{\uparrow} \in \beta(X,\leqslant)$. Let $\rho = \rho_{(X,\leqslant)} : \beta(X,=) \to \beta(X,\leqslant)$ be the map which sends $\mathcal{F} \in \beta(X,=)$ to $\mathcal{F}^{\uparrow}$. Since, for $Z \in \alpha X, \rho^{-1}(Z^{\sharp,\leqslant}) = Z^{\sharp,=}$, ρ is continuous.

Lemma 5. *For each poset $(X,\leqslant)$, $\rho : \beta(X,=) \to \beta(X,\leqslant)$ is surjective.*

Proof. Assume $\mathcal{F} \in \beta(X,\leqslant)$. Define $\mathcal{F}_0 = \{A\backslash B \mid A \in \mathcal{F},\ B \in \alpha_{(X,\leqslant)}\backslash\mathcal{F}\}$. Since $X = X\backslash\emptyset \in \mathcal{F}_0$ and $A_1\backslash B_1,\ A_2\backslash B_2 \in \mathcal{F}_0$ implies $(A_1 \cap A_2)\backslash(B_1 \cup B_2) \in \mathcal{F}_0$, $\mathcal{F}_0$ is a filter base. Extend $\mathcal{F}_0$ to an ultrafilter $\widehat{\mathcal{F}}$. Clearly $\rho(\widehat{\mathcal{F}}) = \mathcal{F}$. $\square$

Theorem 6. *$\beta(X,\leqslant)$ is a Priestley space.*

Proof. $\beta(X,\leqslant)$ is compact because it is the continuous image of the compact space $\beta(X,=)$. To see that $\beta(X,\leqslant)$ is totally-order separated, suppose $\mathcal{F} \not\subseteq \mathcal{G}$. Choose Z an upper set with $Z \in \mathcal{F}\backslash\mathcal{G}$. Then $Z^{\sharp,\leqslant}$ is a clopen upper set which contains $\mathcal{F}$ but not $\mathcal{G}$. $\square$

Define a preorder on $\beta(X,=)$ by the rule: $\mathcal{F} \preceq \mathcal{G}$ if $\rho\mathcal{F} \subseteq \rho\mathcal{G}$. For a poset Y and a function $f : \beta(X,=) \to Y$, we will say that f is *monotone* if whenever $\mathcal{F} \preceq \mathcal{G}$ in $\beta(X,=)$, $f(\mathcal{F}) \leqslant f(\mathcal{G})$ in Y.

Lemma 7. *For any partially ordered topological space Y and monotone continuous function $f : \beta(X,=) \to Y$, there is a unique monotone continuous function $f^{\sim} : \beta(X,=) \to Y$ satisfying $f^{\sim}(\rho\mathcal{F}) = f(\mathcal{F})$ for all $\mathcal{F} \in \beta(X,=)$:*

$$\begin{tikzcd} \beta(X,=) \arrow[r, "\rho"] \arrow[dr, "f"'] & \beta(X,\leqslant) \arrow[d, dotted, "f^{\sim}"] \\ & Y. \end{tikzcd}$$

Proof. Since $\rho_{(X,\leqslant)} : \beta(X,=) \to \beta(X,\leqslant)$ is continuous, the patch topology is T_2 and $\beta(X,=)$ is compact, $\rho_{(X,\leqslant)}$ is a quotient map. The result follows at once from this observation and the universal property of quotient maps. $\square$

For a function $f : X \to Y$, define $\beta(f,=) : \beta(X,=) \to \beta(Y,=)$ by $\beta(f,=)(\mathcal{F}) = \{Z \subseteq Y \mid f^{-1}(Z) \in \mathcal{F}\}$. Since $\beta(f,=)^{-1}(Z^{\sharp,=}) = (f^{-1}(Z))^{\sharp,=}$, $\beta(f,=)$ is continuous. If X and Y are posets and f is monotone, define $\beta(f,\leqslant) : \beta(X,\leqslant) \to \beta(Y,\leqslant)$ by $\beta(f,\leqslant)(\mathcal{F}) = \{Z \in \alpha Y \mid f^{-1}(Z) \in \mathcal{F}\}$. Clearly $\beta(f,\leqslant)$ is monotone and continuous.

Also $\rho_Y \circ \beta(f,=)$ is easily seen to be monotone as well and so $\beta(f,\leqslant)$ is the unique monotone continuous map $\beta(f,\leqslant):\beta(X,\leqslant) \to \beta(Y,\leqslant)$ making the diagram

$$\begin{tikzcd} \beta(X,=) \arrow[r, "\rho_X"] \arrow[d, "\beta(f,=)"'] & \beta(X,\leqslant) \arrow[d, dotted, "\beta(f,\leqslant)"] \\ \beta(Y,=) \arrow[r, "\rho_Y"] & \beta(Y,\leqslant) \end{tikzcd}$$

commute. Evidently, $\beta(\mathrm{id}_X,\leqslant) = \mathrm{id}_{\beta(X,\leqslant)}$ and $\beta(g \circ f,\leqslant) = \beta(g,\leqslant) \circ \beta(f,\leqslant)$. We therefore obtain a functor $\beta:\mathbf{PoSet} \to \mathbf{CmptPoSp}$.

Lemma 8. *Assume that X is a partially ordered topological space. Then X is a pospace iff whenever $\mathcal{F}$ is an ultrafilter on X, $u \in \text{clu\_pt}(\mathcal{F}^{\downarrow})$, and $v \in \text{clu\_pt}(\mathcal{F}^{\uparrow})$, then $u \leqslant v$.*

Proof. Suppose that whenever $\mathcal{F}$ is an ultrafilter on X,

$$u \in \text{clu\_pt}(\mathcal{F}^{\downarrow}), \quad \text{and} \quad v \in \text{clu\_pt}(\mathcal{F}^{\uparrow}),$$

then $u \leqslant v$. We wish to show that $\leqslant$ is closed, so assume (u,v) is in the closure of $\leqslant$; that is, for all $U \in \mathcal{N}(u)$ and $V \in \mathcal{N}(v)$, $(U \times V) \cap \leqslant \neq \emptyset$. Then the sets $\{\uparrow U \cap \downarrow V \mid U \in \mathcal{N}(u) \text{ and } V \in \mathcal{N}(v)\}$ are all nonempty and so they generate a filter, which can be extended to an ultrafilter, $\mathcal{F}$. For K in $\mathcal{F}$ and $U \in \mathcal{N}(u)$, $\uparrow U \cap K \neq \emptyset$, since $\uparrow U \in \mathcal{F}$. Thus U meets every lower set in $\mathcal{F}$. It follows that $u \in \text{clu\_pt}(\mathcal{F}^{\downarrow})$. Similarly, $v \in \text{clu\_pt}(\mathcal{F}^{\uparrow})$. Hence $u \leqslant v$.

Conversely, suppose that X is a pospace. Assume $\mathcal{F}$ is an ultrafilter on X, $u \in \text{clu\_pt}(\mathcal{F}^{\downarrow})$, and $v \in \text{clu\_pt}(\mathcal{F}^{\uparrow})$. If $U \in \mathcal{N}(u)$, then for any $K \in \mathcal{F}$, $U \cap \downarrow K \neq \emptyset$ and so $\uparrow U \cap K \neq \emptyset$. It follows that $\uparrow U \in \mathcal{F}$. Similarly, for all $V \in \mathcal{N}(v)$, $\downarrow V \in \mathcal{F}$. Consequently for all $U \in \mathcal{N}(u)$ and $V \in \mathcal{N}(v)$, $\uparrow U \cap \downarrow V \neq \emptyset$ and so $(U \times V) \cap \leqslant \neq \emptyset$. Since X is a pospace, $u \leqslant v$. $\square$

Proposition 9. *Assume X is a compact pospace. Then every ultrafilter on X has a unique limit and the map $\lim:\beta(X,=) \to X$ which sends an ultrafilter to its unique limit is monotone and continuous.*

Proof. Since X is compact, every ultrafilter on X has a limit point. Uniqueness follows easily from Lemma 8.

Suppose $\mathcal{F} \preceq \mathcal{G}$ in $\beta(X,=)$, $u = \lim \mathcal{F}$ and $v = \lim \mathcal{G}$. Then $u \in \text{clu\_pt}(\mathcal{F}^{\downarrow})$ and $v \in \text{clu\_pt}(\mathcal{G}^{\uparrow})$. But $\mathcal{F}^{\uparrow} \subseteq \mathcal{G}^{\uparrow}$, so $v \in \text{clu\_pt}(\mathcal{F}^{\uparrow})$. By Lemma 8, $u \leqslant v$. It follows that lim is monotone as a map from $\beta(X,=)$ to X. Suppose that U is open in X and $\lim \mathcal{F} \in U$. Choose V open in X so that $\lim \mathcal{F} \in V \subseteq \mathrm{cl}(V) \subseteq U$. Then $V^{\sharp,=}$ is open in $\beta(X,=)$, $\mathcal{F} \in V^{\sharp,=}$ and $V^{\sharp,=} \subseteq \lim^{-1} U$. Hence lim is continuous. $\square$

For X a compact pospace, let $\xi = \xi_{(X,\leqslant)} : \beta(X,\leqslant) \to X$ be the unique monotone continuous map satisfying $\xi(\rho\mathcal{F}) = \lim \mathcal{F}$ for all $\mathcal{F} \in \beta(X,=)$, which is guaranteed to exist by Proposition 9 and Lemma 7:

$$\begin{tikzcd} \beta(X,=) \arrow[r, "\rho"] \arrow[dr, "\lim"'] & \beta(X,\leqslant) \arrow[d, dotted, "\xi"] \\ & X. \end{tikzcd}$$

Define $\eta = \eta_{(X,\leqslant)} : X \to \beta(X,\leqslant)$ by $\eta(x) = \{Z \in \alpha X \mid x \in Z\}$. η is clearly monotone. The following theorem justifies our notation $\beta(X,\leqslant)$ for $\mathsf{Spec}\,\alpha(X,\leqslant)$, since it shows that $\beta(X,\leqslant)$ is the Stone–Čech–Nachbin order compactification [9] of the corresponding discrete ordered space.

Theorem 10. *For any compact pospace Y and any monotone map $f : X \to Y$ there is a unique monotone continuous map $\hat{f} : \beta(X,\leqslant) \to Y$ such that $\hat{f} \circ \eta = f$:*

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & \beta(X,\leqslant) \arrow[d, dotted, "\hat{f}"] \\ & Y \end{tikzcd}$$

Proof. Consider a nonempty basic open subset $Z^{\sharp,\leqslant} \backslash W^{\sharp,\leqslant}$ of $\beta(X,\leqslant)$. Then $Z \not\subseteq W$ and we can choose a point $x \in Z \backslash W$. Then $\eta(x) \in Z^{\sharp,\leqslant} \backslash W^{\sharp,\leqslant}$. It follows that $\eta[X]$ is dense in $\beta(X,\leqslant)$ and so $\hat{f}$ is unique, if it exists. For existence, let $\hat{f} = \xi_Y \circ \beta(f,\leqslant)$. Then $\hat{f}$ is monotone and continuous, and for all $x \in X$,

$$\begin{aligned} \hat{f}\big(\eta_X(x)\big) &= \xi_Y\big(\beta(f,\leqslant)(\eta_X(x))\big) = \xi_Y\big(\eta_Y(f(x))\big) \\ &= \lim\big\{B \subseteq Y \mid f(x) \in B\big\} = f(x). \qquad \square \end{aligned}$$

The forgetful functor $G : \mathbf{CmptPoSp} \to \mathbf{PoSet}$ sends a compact pospace $(X,\tau,\leqslant)$ to the underlying poset $(X,\leqslant)$ and sends a monotone continuous map $f : (X,\tau,\leqslant) \to (Y,\tau,\leqslant)$ to the same function regarded as a monotone map from $(X,\leqslant)$ to $(Y,\leqslant)$. By Theorem 10, $\beta(\cdot,\leqslant)$ is left-adjoint to G and η is the unit and ξ the counit of this adjunction. Thus $\beta(X,\leqslant)$ is the free compact pospace on the poset X. This adjunction induces a monad $\boldsymbol{B} = (\beta,\eta,\mu)$ on **PoSet**, where $\beta = G \circ \beta(\cdot,\leqslant)$. Thus βX is the collection of prime filters on αX partially ordered by inclusion. For $\phi \in \beta(\beta(X,\leqslant),\leqslant)$, $\mu_X(\phi) = \lim \psi$, where $\psi \in \beta(\beta(X,\leqslant),=)$ satisfies $\rho_{\beta(X,\leqslant)}\psi = \phi$. Since for $Z \in \alpha X$, $Z^{\sharp,\leqslant}$ is clopen in $\beta(X,\leqslant)$,

$$Z \in \mu_X(\phi) \Leftrightarrow Z \in \lim \psi \Leftrightarrow \lim \psi \in Z^{\sharp,\leqslant} \Leftrightarrow Z^{\sharp,\leqslant} \in \psi \Leftrightarrow Z^{\sharp,\leqslant} \in \phi.$$

We wish to show that the forgetful functor $G : \mathbf{CmptPoSp} \to \mathbf{PoSet}$ is monadic; that is, the comparison functor $K : \mathbf{CmptPoSp} \to \mathbf{PoSet}^{\boldsymbol{B}}$ is an isomorphism of categories. For this we will adapt Manes' [6] original proof as presented in Johnstone [2, §III.2], of the corresponding result for compact Hausdorff spaces, to the setting of compact pospaces.

Wading through the definitions, we see that a $\boldsymbol{B}$-algebra consists of a poset X together with a monotone map $\gamma : \beta X \to X$ such that the following two conditions hold:

($\beta 1$) for all $x \in X$, $\gamma(\eta(x)) = x$; and

($\beta 2$) for all $\phi \in \beta(\beta(X, \leqslant), \leqslant)$, $\gamma\{Z \in \alpha X \mid Z^{\sharp,\leqslant} \in \phi\} = \gamma\{Z \in \alpha X \mid \gamma^{-1}(Z) \in \phi\}$.

Lemma 11. *Assume X and Y are compact pospaces and $f : X \to Y$ is a monotone map. Then f is continuous iff f is a morphism of the $\boldsymbol{B}$-algebras $(X, \gamma_X : \beta X \to X)$ and $(Y, \gamma_Y : \beta Y \to Y)$.*

Proof. f is a morphism of the $\boldsymbol{B}$-algebras $(X, \gamma : \beta X \to X)$ and $(Y, \gamma : \beta Y \to Y)$ iff the diagram

$$\begin{tikzcd} \beta X \arrow[r, "\beta f"] \arrow[d, "\gamma_X"'] & \beta Y \arrow[d, "\gamma_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes; that is, for all ultrafilters $\mathcal{F}$ on X, $f(\gamma_X(\rho\mathcal{F})) = \gamma_Y(\beta f(\rho\mathcal{F}))$. From the definitions of βf and γ, this is equivalent to the condition that f preserve limits of ultrafilters, which is equivalent to continuity. $\square$

Theorem 12. *The forgetful functor G :* **PoSet** $\to$ **CmptPoSp** *is monadic.*

Proof. By Lemma 11, the comparison functor K : **CmptPoSp** $\to$ $\mathbf{PoSet}^{\boldsymbol{B}}$, which sends the compact pospace X to the $\boldsymbol{B}$-algebra $((X, \leqslant), \gamma : \beta(X, \leqslant) \to (X, \leqslant))$, where $\gamma\rho\mathcal{F} = \lim \mathcal{F}$ for all $\mathcal{F} \in \beta(X, =)$, is full and faithful. So all we need to do is to show that every $\boldsymbol{B}$-algebra structure on a poset $(X, \leqslant)$ is induced in this way by a (necessarily unique) topology τ on X which makes $(X, \tau, \leqslant)$ a compact pospace.

Let (X, γ) be a $\boldsymbol{B}$-algebra and define for $A \subseteq X$,

$$\overline{A} = \{\gamma(\rho\mathcal{F}) \mid A \in \mathcal{F} \in \beta(X, =)\}.$$

We will show that $A \mapsto \overline{A}$ defines a Kuratowski closure operator on X. From ($\beta 1$) it follows at once that $A \subseteq \overline{A}$. Trivially, $A \subseteq B$ implies that $\overline{A} \subseteq \overline{B}$. Suppose that $x \in \overline{A \cup B}$. Choose $\mathcal{F}$ an ultrafilter such that $(A \cup B) \in \mathcal{F}$ and $\gamma(\rho\mathcal{F}) = x$. Then $A \in \mathcal{F}$ or $B \in \mathcal{F}$, so $x \in \overline{A} \cup \overline{B}$. It remains to show that $\overline{\overline{A}} \subseteq \overline{A}$. Suppose $x \in \overline{\overline{A}}$. Choose $\mathcal{F}$ an ultrafilter on X so that $\overline{A} \in \mathcal{F}$ and $\gamma(\rho\mathcal{F}) = x$. For each $B \in \mathcal{F}$, $B \cap \overline{A} \neq \emptyset$, and so $\rho^{-1}\gamma^{-1}(B) \cap A^{\sharp,=} \neq \emptyset$. Consequently the family $\{\rho^{-1}_{(X,\leqslant)}\gamma^{-1}(B) \cap A^{\sharp,=} \mid B \in \mathcal{F}\}$ can be extended to a ultrafilter ϕ on $\beta(X, =)$. But then $\{Z \subseteq X \mid \rho^{-1}\gamma^{-1}(Z) \in \phi\} = \mathcal{F}$ and $A \in \{Z \subseteq X \mid Z^{\sharp,=} \in \phi\}$. Let

$$\psi = \{\mathcal{A} \in \alpha\beta(X, \leqslant) \mid \rho^{-1}\mathcal{A} \in \phi\}.$$

Then $\psi \in \beta(\beta(X, \leqslant), \leqslant)$,

$$\{Z \in \alpha X \mid \gamma^{-1}(Z) \in \psi\} = \rho_{(X,\leqslant)}\{Z \subseteq X \mid \rho^{-1}\gamma^{-1}(Z) \in \phi\} = \rho\mathcal{F},$$

and

$$\{Z \in \alpha X \mid Z^{\sharp,\leqslant} \in \psi\} = \rho\{Z \subseteq X \mid Z^{\sharp,=} \in \phi\}.$$

By ($\beta 2$), $\gamma(\{Z \in \alpha X \mid Z^{\sharp,\leqslant} \in \psi\}) = x$ and so $x \in \overline{A}$.

Thus $\{A \subseteq X \mid A = \overline{A}\}$ is the lattice of closed sets of a topology, τ, on X. For $\mathcal{F}$ an ultrafilter on X, $\gamma(\rho\mathcal{F})$ is in $\overline{A}$ for every $A \in \mathcal{F}$, so $\gamma(\rho\mathcal{F})$ is a limit point of $\mathcal{F}$. It follows that (X,τ) is compact. To see that $(X,\tau,\leqslant)$ is a pospace, suppose that $\mathcal{F}$ is an ultrafilter on X, $u \in \text{clu\_pt}(\mathcal{F}^{\downarrow})$, and $v \in \text{clu\_pt}(\mathcal{F}^{\uparrow})$. Let $x = \gamma(\rho\mathcal{F})$. Then for each $B \in \mathcal{F}$, $v \in \overline{\uparrow B}$ so there is an ultrafilter $\mathcal{G}$ on X such that $\uparrow B \in \mathcal{G}$ and $\gamma(\rho\mathcal{G}) = v$. Consequently, for each $B \in \mathcal{F}$, $\rho^{-1}\gamma^{-1}(v) \cap (\uparrow B)^{\sharp,=} \neq \emptyset$ and so this family of sets can be extended to an ultrafilter, ϕ, on $\beta(X,=)$. Since $\rho^{-1}\gamma^{-1}(v) \in \phi$,

$$\{Z \subseteq X \mid \rho^{-1}\gamma^{-1}(Z) \in \phi\} = \{Z \subseteq X \mid v \in Z\}.$$

Also $\mathcal{F}^{\uparrow} \subseteq \{Z \subseteq X \mid Z^{\sharp,=} \in \phi\}$. Let

$$\psi = \{\mathcal{A} \in \alpha\beta(X,\leqslant) \mid \rho^{-1}\mathcal{A} \in \phi\}.$$

Then $\psi \in \beta(\beta(X,\leqslant),\leqslant)$,

$$\{Z \in \alpha X \mid \gamma^{-1}(Z) \in \psi\} = \rho\{Z \subseteq X \mid \rho^{-1}\gamma^{-1}(Z) \in \phi\} = \eta(v),$$

and

$$\{Z \in \alpha X \mid Z^{\sharp,\leqslant} \in \psi\} = \rho\{Z \subseteq X \mid Z^{\sharp,=} \in \phi\}.$$

By ($\beta 1$), ($\beta 2$) and the monotonicity of γ,

$$\begin{aligned} x = \gamma\rho\mathcal{F} &\leqslant \gamma\{Z \in \alpha X \mid Z^{\sharp,\leqslant} \in \psi\} = \gamma\{Z \in \alpha X \mid \gamma^{-1}(Z) \in \psi\} \\ &= \gamma(\eta(v)) = v. \end{aligned}$$

A dual argument shows that $u \leqslant x$. Thus $u \leqslant v$. Consequently $(X,\tau,\leqslant)$ is a compact pospace, which induces the $\boldsymbol{B}$-algebra (X,γ). $\square$

The following commutative diagram may give the reader some insight into the proof of Theorem 12:

$$\begin{tikzcd}
\beta(\beta(X,=),=) \arrow[rr, "\mu_{(X,=)}"] \arrow[d, "\beta(\rho_X,=)"'] & & \beta(X,=) \arrow[d, "\rho_X"] \\
\beta(\beta(X,\leqslant),=) \arrow[r, "\rho_{\beta(X,\leqslant)}"] \arrow[d, "\beta(\gamma,=)"'] & \beta(\beta(X,\leqslant),\leqslant) \arrow[r, "\mu_{(X,\leqslant)}"] \arrow[d, "\beta(\gamma,\leqslant)"'] & \beta(X,\leqslant) \arrow[d, "\gamma"] \\
\beta(X,=) \arrow[r, "\rho_X"'] & \beta(X,\leqslant) \arrow[r, "\gamma"'] & X.
\end{tikzcd}$$

Here $\mu_{(X,=)}$ is the multiplication of the ultrafilter monad and is defined by

$$Z \in \mu_{(X,=)}(\phi) \Leftrightarrow Z^{\sharp,=} \in \phi.$$

Thus the proof could be reformulated in terms of the morphism of monads ρ between the ultrafilter monad and the prime upper filter monad. We have avoided this reformulation in order to keep the argument elementary.

Example 13. We now find $\beta(X,\leqslant)$ in two examples: when X is the algebraic poset $(\omega,\leqslant)$ and when X is the algebraic lattice $(\omega\cup\{\infty\},\leqslant)$, with $\leqslant$ in both cases the usual order.

In either case, consider two free ultrafilters on X, $\mathcal{F},\mathcal{G}$. If $K\in\mathcal{F}$, then K is infinite, thus $\uparrow K$ is a cofinite subset of X, so $\uparrow K\in\mathcal{G}$. As a result of the arbitrary nature of $\mathcal{F},\mathcal{G}$ all free ultrafilters equivalent with respect to $\preceq\cap\preceq^{-1}$. Thus in both cases, $\beta(X,\leqslant)$ has just one point outside of X, which we call L. Again, in either case, notice that for each $n\in\omega$, $\eta(n)\preceq\mathcal{F}$ for each free ultrafilter $\mathcal{F}$, since if $n\in K$ then $\uparrow K$ is cofinite, and so in $\mathcal{F}$. Also, in the second case, if $\mathcal{G}$ is any ultrafilter, then $\mathcal{G}\preceq\eta(\infty)$, since ∞ is in every nonempty upper set.

Thus $\beta(\omega,\leqslant)$ is order-isomorphic to $\omega\cup\{\infty\}$, and $\beta(\omega\cup\{\infty\},\leqslant)$ is order-isomorphic to $\omega\cup\{L,\infty\}$, with $n<L<\infty$ for each $n\in\omega$. For the topologies, notice that for each point in $x\in X$, $\eta(x)$ is isolated, since $\uparrow x^{\sharp,\leqslant}\setminus\uparrow(x+1)^{\sharp,\leqslant}=\{\eta(x)\}$. As a result, by the compactness of $\beta(X,\leqslant)$, $\{L\}$ is the limit of all nonfixed ultrafilters.

4. The prime open filter monad

Simmons [11] and Wyler [14] have shown that that the category of compact pospaces with continuous monotone maps is also algebraic over the category of topological spaces and continuous maps. Here the "forgetful functor" $G^{\boldsymbol{W}}:\mathbf{CmptPoSp}\to\mathbf{Sp}$ sends the compact pospace $(X,\tau,\leqslant)$ to $(X,\tau^{\uparrow})$ and the corresponding monad on $\mathbf{Sp}$ is the *prime open filter monad*: $\boldsymbol{W}=(\omega,\theta,\nu)$, where

- for a topological space (X,τ), $\omega(X,\tau)$ is the set of prime filters on τ with the spectral topology;
- for $f:(X,\tau_X)\to(Y,\tau_Y)$ a continuous map, $\omega(f):\omega(X,\tau)\to\omega(Y,\tau)$ is defined by $\omega(f)(\mathcal{F})=\{U\in\tau_Y\mid f^{-1}(U)\in\mathcal{F}\}$, for $\mathcal{F}\in\omega(X,\tau_X)$;
- $\theta_X:X\to\omega(X)$ sends $x\in X$ to $\{U\in\tau_X\mid x\in U\}$; and
- $\nu_X:\omega^2(X)\to\omega(X)$ sends $\phi\in\omega^2(X)$ to $\{U\in\tau_X\mid U^{\sharp,\tau}\in\phi\}$, where $U^{\sharp,\tau}=\{\mathcal{F}\in\omega(X)\mid U\in\mathcal{F}\}$.

We omit the simple verification that $\boldsymbol{W}$ is indeed a monad on $\mathbf{Sp}$ and instead use Theorem 12 to show that $\mathbf{Sp}^{\boldsymbol{W}}$ is isomorphic to the category of compact pospaces with continuous monotone maps.

Let $R:\mathbf{Sp}\to\mathbf{Poset}$ be the functor that sends a topological space (X,τ) to the poset $(X,\leqslant_\tau)$ and sends a continuous map $f:(X,\tau_X)\to(Y,\tau_Y)$ to itself. Since continuous maps are monotone for the specialization orders, R is indeed a functor. For a topological space (X,τ), let

$$\pi_X:\beta(X,\leqslant_\tau)\to(\omega(X,\tau),\subseteq)$$

be the monotone map which sends $\mathcal{F} \in \beta(X, \leqslant_\tau)$ to $\mathcal{F} \cap \tau$. Then $\pi : \beta(\cdot, \leqslant) \circ R \to R \circ \omega(\cdot, \tau)$ is a natural transformation and simple calculation shows that $(R, \pi) : \boldsymbol{W} \to \boldsymbol{B}$ is a morphism of monads. We therefore obtain a functor $(R, \pi)^\star : \mathbf{Sp}^{\boldsymbol{W}} \to \mathbf{Poset}^{\boldsymbol{B}}$ such that diagram

$$\begin{tikzcd} \mathbf{Sp} \arrow[d, "R"] & \mathbf{Sp}^{\boldsymbol{W}} \arrow[l, "G^{W}"'] \arrow[d, "(R,\pi)^\star"] \\ \mathbf{Poset} & \mathbf{Poset}^{\boldsymbol{B}} \arrow[l, "G^{B}"] \end{tikzcd}$$

commutes. We will show that $(R, \pi)^\star$ is an isomorphism of categories.

Lemma 14. *For any topological space (X, τ), the map $\pi_X : \beta(X, \leqslant_\tau) \to (\omega(X, \tau), \subseteq)$ is surjective.*

Proof. Assume $\mathcal{F} \in \omega(X, \tau)$. Define $\mathcal{F}_0 = \{U \backslash V \mid U \in \mathcal{F},\ V \in \tau \backslash \mathcal{F}\}$. Then $\mathcal{F}_0$ is a filter base which can be extended to an ultrafilter $\widehat{\mathcal{F}}$. Clearly $\pi_X(\rho_X(\widehat{\mathcal{F}})) = \mathcal{F}$. $\square$

Lemma 15. *For any topological space (X, τ), the map*

$$\beta\pi_X : \beta^2(X, \leqslant_\tau) \to \beta\big(\omega(X, \tau), \subseteq\big)$$

is surjective.

Proof. Since the diagram

$$\begin{tikzcd} \beta(\beta(X, \leqslant_\tau), =) \arrow[r, "\beta(\pi_X, =)"] \arrow[d, "\rho_{\beta(X, \leqslant_\tau)}"'] & \beta(\omega(X, \tau), =) \arrow[d, "\rho_{\omega(X, \tau)}"] \\ \beta(\beta(X, \leqslant_\tau), \subseteq) \arrow[r, "\beta(\pi_X, \leqslant)"'] & \beta(\omega(X, \tau), \subseteq) \end{tikzcd}$$

commutes and $\rho_{\omega(X,\tau)}$ is surjective, it will suffice to show that $\beta(\pi_X, =)$ is surjective. So let $\phi \in \beta(\omega(X, \tau), =)$. Then $\{\pi^{-1}Z \mid Z \in \phi\}$ is a filter base on $\beta(X, \leqslant_\tau)$ which can be extended to an ultrafilter $\psi \in \beta(\beta(X, \leqslant_\tau), =)$. Then

$$\beta(\pi_X, =)\psi = \{Z \subseteq \omega(X, \tau) \mid \pi^{-1}Z \in \psi\} \supseteq \phi.$$

But $\beta(\pi_X, =)\psi$ is an ultrafilter on $\omega(X, \tau)$, so $\beta(\pi_X, =)\psi = \phi$. $\square$

Lemma 16. *Assume $(X, \tau, \leqslant)$ is a compact pospace, $\mathcal{F}, \mathcal{G} \in \beta(X, =)$ and $\mathcal{F} \cap \tau^\uparrow \subseteq \mathcal{G}$. Then* $\lim \mathcal{F} \leqslant \lim \mathcal{G}$.

Proof. Suppose $\lim \mathcal{F} \not\leqslant \lim \mathcal{G}$. Choose $U \in \tau^\uparrow$, $V \in \tau^\downarrow$ so that $\lim \mathcal{F} \in U$, $\lim \mathcal{G} \in V$ and $U \cap V = \emptyset$. Then $U \in \mathcal{F} \cap \tau^\uparrow$ and so $U \in \mathcal{G}$. Hence $V \notin \mathcal{G}$, which is absurd. $\square$

Lemma 17. *Assume $(X, \tau, \leqslant)$ is a compact pospace and let $h : \beta(X, \leqslant) \to X$ be the corresponding $\boldsymbol{B}$-algebra structure on $(X, \leqslant)$. Then there is a unique continuous map*

$\bar{h}:\omega(X,\tau^{\uparrow}) \to (X,\tau^{\uparrow})$ *such that* $h = \bar{h} \circ \pi_{(X,\tau^{\uparrow})}$. *Moreover,* $((X,\tau^{\uparrow}),\bar{h})$ *is a* $\boldsymbol{W}$-*algebra.*

Proof. For any $\mathcal{F} \in \beta(X,=)$, $h(\rho_X\mathcal{F}) = \lim_\tau \mathcal{F}$. Moreover, $h:\beta(X,\leqslant) \to X$ is continuous and monotone. Since

$$\pi^{-1}_{(X,\tau^{\uparrow})}\big(U^{\sharp,\tau}\big) = U^{\sharp,\leqslant}, \quad \text{for} \quad U \in \tau^{\uparrow},$$

$\pi_{(X,\tau^{\uparrow})}:\beta(X,\leqslant) \to \omega(X,\tau^{\uparrow})$ is continuous for the patch topologies. But these spaces are compact and Hausdorff and $\pi_{(X,\tau^{\uparrow})}$ is surjective, so $\pi_{(X,\tau^{\uparrow})}$ is actually a quotient map. It follows that there is a unique continuous map $\bar{h}:\omega(X,\tau^{\uparrow}) \to (X,\tau^{\uparrow})$ such that $h = \bar{h} \circ \pi_{(X,\tau^{\uparrow})}$.

From the Lemma 16 it follows that $\bar{h}$ is monotone and so continuous as a map

$$\omega(X,\tau^{\uparrow}) \to (X,\tau^{\uparrow}).$$

Trivially, $\bar{h}\eta^{(\boldsymbol{W})}_X(x) = x$. To see that $\bar{h} \circ \mu^{(\boldsymbol{W})}_X = \bar{h} \circ \omega(\bar{h})$, consider the diagram

$$\begin{tikzcd}
\beta^2(X,\leqslant) \arrow[rr, "\mu^{(\boldsymbol{B})}_{(X,\leqslant)}"] \arrow[d, "\beta(\pi_X,\leqslant)"'] & & \beta(X,\leqslant) \arrow[d, "\pi_{(X,\tau^{\uparrow})}"] \\
\beta(\omega(X,\tau^{\uparrow}),\subseteq) \arrow[r, "\pi_{\omega(X,\tau^{\uparrow})}"] \arrow[d, "\beta(\bar{h},\leqslant)"'] & \omega^2(X,\tau^{\uparrow}) \arrow[r, "\mu^{(\boldsymbol{W})}_{(X,\tau^{\uparrow})}"] \arrow[d, "\omega\bar{h}"'] & \omega(X,\tau^{\uparrow}) \arrow[d, "\bar{h}"] \\
\beta(X,\leqslant) \arrow[r, "\pi_{(X,\tau^{\uparrow})}"'] & \omega(X,\tau^{\uparrow}) \arrow[r, "\bar{h}"'] & (X,\tau^{\uparrow})
\end{tikzcd}$$

The outer square commutes because (X,h) is an $\boldsymbol{B}$-algebra. The upper square commutes because $\pi:\boldsymbol{B} \to \boldsymbol{W}$ is a morphism of monads. The lower left square commutes because $\pi:\beta(\cdot,\leqslant) \circ R \to R \circ \omega$ is a natural transformation. Since $\beta(\pi_{(X,\tau^{\uparrow})},\leqslant)$ and $\pi_{(X,\tau^{\uparrow})}$ are surjective, the rectangle on the lower right also commutes. Hence $((X,\tau^{\uparrow}),\bar{h})$ is a $\boldsymbol{W}$-algebra. □

If $f:(X,h) \to (Y,i)$ is a morphism of $\boldsymbol{B}$-algebras (X,h) and (Y,i), then f is continuous and monotone. Hence $f:(X,\tau^{\uparrow}) \to (Y,\tau^{\uparrow})$ is continuous. Consider the diagram:

$$\begin{tikzcd}
\beta(X,\leqslant) \arrow[r, "\beta(f,\leqslant)"] \arrow[d, "\pi_{(X,\tau^{\uparrow}_X)}"] & \beta(Y,\tau^{\uparrow}) \arrow[d, "\pi_{(Y,\tau^{\uparrow}_Y)}"] \\
\omega(X,\tau^{\uparrow}_X) \arrow[r, "\omega f"] \arrow[d, "\bar{h}"] & \omega(Y,\tau^{\uparrow}_Y) \arrow[d, "\bar{i}"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The outer rectangle commutes because f is a morphism of $\boldsymbol{B}$-algebras and the upper rectangle commutes because π is natural. Since $\pi_{(X,\tau^{\uparrow}_X)}$ is surjective, the lower rectangle also commutes. Hence f is a morphism of the $\boldsymbol{W}$-algebras $((X,\tau^{\uparrow}),\bar{h})$ and $((Y,\tau^{\uparrow}),\bar{i})$.

We obtain a functor $H:\mathbf{Poset}^{\boldsymbol{B}} \to \mathbf{Sp}^{\boldsymbol{W}}$. We claim that H is inverse to $(R,\pi)^\star$. For a compact pospace $(X, h:\beta(X,\leqslant) \to X)$, since $HX = ((X,\tau^\uparrow), \bar{h}:\omega(X,\tau^\uparrow) \to X)$, $\leqslant_{\tau^\uparrow} = \leqslant$ and by construction, $h = \bar{h}\circ\pi_X$. Consequently, $(R,\pi)^\star \circ H = \mathrm{id}_{\mathbf{Poset}^{\boldsymbol{B}}}$. For a $\boldsymbol{W}$-algebra $((X,\tau), k:\omega(X,\tau) \to (X,\tau))$, let

$$(R,\pi)^\star\big((X,\tau),k\big) = (X,\tau^s,\leqslant_\tau) \quad \text{and} \quad h = k\circ\pi_X : \beta(X,\leqslant_\tau) \to (X,\leqslant_\tau).$$

Then the diagram

$$\begin{tikzcd}
\beta(X,=) \arrow[r, "\lim_{\tau^s}"] \arrow[d, "\rho_{(X,\leqslant_\tau)}"'] & (X,\tau^s) \arrow[d, "\mathrm{id}"] \\
\beta(X,\leqslant_\tau) \arrow[r, "h"] \arrow[d, "\pi_{(X,\tau)}"'] & (X,\leqslant_\tau) \arrow[d, "\mathrm{id}"] \\
\omega(X,\tau) \arrow[r, "k"] & (X,\tau)
\end{tikzcd}$$

commutes. Since k, $\rho_{(X,\leqslant_\tau)}$ and $\pi_{(X,\tau)}$ are continuous and $\lim_{\tau^s}$ is a quotient map, $\mathrm{id}:(X,\tau^s) \to (X,\tau)$ is continuous. Thus $\tau \subseteq \tau^s$. From this it follows that for $\mathcal{F} \in \beta(X,=)$ if $x \leqslant_\tau \lim_{\tau^s}\mathcal{F}$, then $\mathcal{F} \to_\tau x$. Conversely, suppose that $\mathcal{F} \to_\tau x$, then

$$\eta^{(\boldsymbol{W})}(x) \subseteq \pi_{(X,\tau)}\rho_{(X,\leqslant_\tau)}\mathcal{F}$$

and so

$$x = k\eta^{(\boldsymbol{W})}(x) \leqslant k\pi_{(X,\tau)}\rho_{(X,\leqslant_\tau)}\mathcal{F} = \lim_{\tau^s}\mathcal{F}.$$

Since also $\mathcal{F} \to_{\tau^{s,\uparrow}} x$ iff $x \leqslant \lim_{\tau^s}\mathcal{F}$, $\tau = \tau^{s,\uparrow}$. It follows that $H(R,\pi)^\star((X,\tau),k) = ((X,\tau),k)$.

Theorem 18 (Simmons [11], Wyler [14]). *The forgetful functor $G^{\boldsymbol{W}}:\mathbf{CmptPoSp} \to \mathbf{Sp}$ is monadic and the category of algebras for the prime open filter monad $\boldsymbol{W}$ on* **Sp** *is isomorphic to the category of compact pospaces and continuous monotone maps.*

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